The Resolvent Method in Complex Dynamics

with a first application on building a bridge between Schramm-Loewner Evolutions and Random Matrix Theory

In just 4 slides

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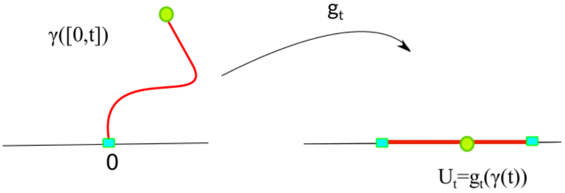
Loewner Differential Equation and SLE



- The maps satisfy the Loewner Differential Equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z\,.$$

- SLE: $U_t = \sqrt{\kappa} B_t$.
- **Brownian Motion** $\to$ **Dyson Brownian Motion**(DBM), $\beta = 8/\kappa$.

$$d\lambda_t^i = \frac{\sqrt{2}}{\sqrt{N\beta}} dB_t^i + \frac{1}{N} \sum_{j \neq i} \frac{dt}{\lambda_t^j - \lambda_t^i}, i = 1, \cdots, N.$$

We consider

$$\partial_t g_t(z) = \frac{1}{N}\sum_{j=1}^{N} \frac{2}{g_t(z) - \lambda_t^j}.$$

Theorem (Campbell-Luh-M.)

Let $\beta = 1$ or $\beta = 2$, and let us consider Dyson Brownian motion beginning at the origin. Let K_T be the multiple SLE hull at time $T > 0$. Then, for any $\varepsilon > 0$, for the multiple SLE maps for N curves, we have that

$$\sup_{t\in[0,T],z\in G} \left| g_t^N(z) - g_t^{\infty}(z) \right| = O\left(\frac{1}{N^{1/3-\varepsilon}}\right),$$

with overwhelming probability, for a given $G \subset \mathbb{H}\backslash K_T$.

- Uses modern RMT techniques such as Stieltjes transforms, self-consistent equations, etc.

Multiple SLE and Random Matrix Theory

$$\partial_t g_t(z) = \frac{1}{N} \sum_{j=1}^{N} \frac{2}{g_t(z) - \lambda_t^j}.$$

- Rewrite RHS of Multiple Loewner equation as $M_t^N(\bullet) = \frac{1}{N} \sum_{j=1}^{N} \frac{2}{\bullet - \lambda_t^j}$.
- Recognize Stieltjes transform for a given choice of measure.
- **Let** $\beta = 1$. **We have that** $M_t^N(\bullet) = \frac{-2}{N} \operatorname{tr} \left(A_t - \bullet \cdot I \right)^{-1}$, **where** $A_t = \sqrt{t} A$, **with** A **a GOE**
- RHS of Loewner can be recovered as a Random Matrix Theory object.
- **The critical parameters** $\kappa = 4$ **and** $\kappa = 8$ **in the SLE theory correspond to the 'nice'** $\beta = 2$ **and** $\beta = 1$ **parameters in the Random Matrix Theory.**

The general Resolvent Method in Complex Dynamics

Let us consider general random or deterministic drivers $\lambda_t^1, \ldots, \lambda_t^N$, and let

$$\partial_t g_t(z) = \frac{1}{N} \sum_{j=1}^{N} \frac{2}{g_t(z) - \lambda_t^j}.$$

- In general, we consider $M_t^N(\bullet) = \frac{1}{N} \sum_{j=1}^{N} \frac{2}{\bullet - \lambda_t^j}$ as $M_t^N(\bullet) = \frac{-2}{N} \operatorname{tr} (A_t - \bullet \cdot I)^{-1}$, where A_t satisfies

$$dA_t = -\frac{1}{2} A_t dt + \frac{1}{\sqrt{N}} dB_t,$$

  or other random or deterministic dynamics in the matrix space.
- **What can we learn?** We propose an alternative way to study Multiple Loewner Dynamics by computing **resolvents** (a nice analytical object) of the dynamics in the matrix space which can be more regular than the drivers (eigenvalues) dynamics.